Tutorial 6

Q.1 (Def. of +ve orientation & physical meaning of the Green's thm)

Let $\mathcal{U} \subseteq \mathbb{R}^2$ be a non-empty open set.

Let $\Omega \subseteq \mathcal{U}$ be a compact domain with connected C^1 boundary $\partial\Omega$.

Let $\alpha : [0,1] \to \partial\Omega$ be a parametrization of $\partial\Omega$ such that $|\alpha'(t)| = 1$ & $\alpha''(t) \neq 0 \quad \forall\, t \in (0,1)$.

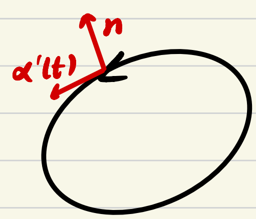
(a) Show that $\alpha'(t) \perp \alpha''(t)$ and $\alpha''(t) \neq 0$. $\forall\, t \in (0,1)$.

(This shows that $\left\{ \frac{\alpha''(t)}{|\alpha''(t)|}, \alpha'(t) \right\}$ form an orthonormal basis (ONB) of $\mathbb{R}^2$.)

Let α define a positive orientation of $\partial\Omega$, i.e. the oriented ONB $\left\{ \frac{\alpha''(t)}{|\alpha''(t)|}, \alpha'(t) \right\}$ is positive oriented as a basis in $\mathbb{R}^2$,

i.e.

$$\det \begin{pmatrix} | & | \\ \frac{\alpha''(t)}{|\alpha''(t)|} & \alpha'(t) \\ | & | \end{pmatrix} > 0, \forall\, t \in (0,1)$$



Let $n = \frac{\alpha''(t)}{|\alpha''(t)|}$. (The outward pointing normal of $\partial\Omega$)

Let $F = (P, Q) : \mathcal{U} \to \mathbb{R}^2$ be a C^1 vector field.

Define the divergence of F as $\operatorname{div}(F) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$

(b) Show that $\displaystyle\int_{\partial\Omega} F \cdot n := \int_0^1 F(\alpha(t)) \cdot n(t)\, dt = \int_{\Omega} \operatorname{div}(F)\, dA$

Solution:

(a) Since $|\alpha'(t)| = 1$, $\alpha'(t) \cdot \alpha'(t) = 1$.

Differentiate both sides w.r.t. t:

$$\alpha''(t) \cdot \alpha'(t) + \alpha'(t) \cdot \alpha''(t) = 0$$

$$\therefore \alpha''(t) \cdot \alpha'(t) = 0$$

$\therefore \alpha'(t) \perp \alpha''(t)$.

(b) Let $\alpha(t) = (x(t), y(t))$.

$$\alpha'(t) = (x'(t), y'(t))$$

Since $\{n, \alpha'(t)\}$ is positively oriented ONB, n is formed by rotating $\alpha'(t)$ clockwise by $90°$, i.e. $n = (y'(t), -x'(t))$.

$$\therefore \int_{\partial\Omega} F \cdot n = \int_0^1 (P y'(t) - Q x'(t))\, dt$$

$$= \int_{\partial\Omega} -Q\,dx + P\,dy$$

$$= \int_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA \quad \text{(Green's thm)}$$

$$= \int_{\Omega} \operatorname{div}(F)\, dA.$$

Remark:

The quantity $\int_{\partial\Omega} F \cdot n$ is called the flux of the vector field.
If we imagine $F(x,y)$ as the direction of flow of a fluid at a point $(x,y) \in U$, then $\int_{\partial\Omega} F \cdot n$ measures the amount of fluid which escapes the region Ω through $\partial\Omega$. (Note that only the component of $F \perp \partial\Omega$ would contribute to this, so we take $F \cdot n$)
Green's thm tells us that this quantity is equivalent to $\int_{\Omega} \operatorname{div}(F) dA$, where $\operatorname{div}(F)(x,y)$ measures the amount of that escapes the point (x,y).
Therefore, the Green's thm can be thought as the conservation of particles: In a "closed system", the particles are not created nor destroyed.

Q.2 (Cauchy - Goursat thm)

Let $f: \mathbb{C} \to \mathbb{C}$ be a complex-valued function.

We identify $\mathbb{C}$ with $\mathbb{R}^2$ through $x+iy \mapsto (x,y)$, so we can write $f(z) = P(x,y) + i\,Q(x,y)$, where $z = x+iy$.

Let $\Omega \subseteq \mathbb{C}$ be a compact domain with C^1 boundary $\partial\Omega$.

Define the complex integral of f as

$$\int_{\partial\Omega} f(z)\,dz = \int_{\partial\Omega} P\,dx - Q\,dy + i\int Q\,dx + P\,dy.$$

Suppose P, Q satisfy the Cauchy–Riemann equation:

$$\begin{cases} P_x = Q_y \\ P_y = -Q_x \end{cases} \quad \text{on some open set } U \supseteq \Omega$$

Show that $\int_{\partial\Omega} f(z)\,dz = 0$.

Informally,

$$\int_{\partial\Omega} f(z)\,dz$$

$$= \int_{\partial\Omega} (P+iQ)(dx+idy)$$

$$= \int_{\partial\Omega} P\,dx - Q\,dy$$

$$+\, i\int_{\partial\Omega} Q\,dx + P\,dy$$

Solution:

$$\int_{\partial\Omega} f(z)\,dz = \int_{\partial\Omega} P\,dx - Q\,dy + i\int Q\,dx + P\,dy.$$

$$= \int_{\Omega} (-Q_x - P_y)\,dA + i\int_{\Omega} (P_x - Q_y)\,dA$$

$= 0$ by the Cauchy-Riemann equation.

Q.3

Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field defined by

$$F(x,y) = (x^2y + y^3 - y + f(x),\ 3x + 2y^2x + g(y))$$

where $f, g: \mathbb{R} \to \mathbb{R}$ are differentiable.

Let $\mathcal{A} := \{ \Omega \subseteq \mathbb{R}^2 : \Omega$ is compact & $\partial\Omega$ is piecewise $C^1 \}$

Find $\Omega \in \mathcal{A}$ which maximizes

$$\int_{\partial\Omega} F \cdot d\vec{r}$$

where $\partial\Omega$ is positively oriented.

Solution:

By the Green's thm,

$$\int_{\partial\Omega} F \cdot d\vec{r} = \int_{\Omega} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dA$$

$$= \int_{\Omega} 3 + 3y^2 - (x^2 + 2y^2 - 1) \, dA$$

$$= \int_{\Omega} (4 - x^2 - y^2) \, dA.$$

$\therefore$ The integral is maximized if

$$\Omega = \{(x, y) : x^2 + y^2 \leq 4\}$$